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Analysis and Sensitivity Evaluation of $2p$-Port Cascaded Networks

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***Abstract*—An exact analysis approach for efficiently evaluating the response and its sensitivities with respect to all design parameters for cascaded $2p$-port networks is presented for any value of p. It is illustrated via a quasi-optical bandpass filter.**

I. Introduction

A GENERALIZATION of an analysis approach for 2-port cascaded networks [1] to handle $2p$-port networks is presented. The generalized approach has the same advantages as those for 2-port networks. These advantages include efficient and fast analytical and numerical investigations of response, first-order sensitivities of the response with respect to variable parameters, and large-change sensitivities. The need for this generalization evolved from the fact that many microwave networks are represented as a cascade of $2p$-port elements.

Thevenin and Norton equivalents for these cascaded networks can be obtained systematically using this approach. These in turn are very useful for worst case analysis [2]. As an example, a quasi-optical bandpass filter has been analyzed using this approach and the exact sensitivities of the response with respect to a parameter appearing in two of the $2p$-port elements, representing the filter elements, have been evaluated.

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II. Theory

The analysis approach consists of two principal types. The first, which we call the forward analysis, consists of initializing a $\bar{U}^T$ matrix as E_1^T or E_2^T, which are defined as

$$E_1 \triangleq \begin{bmatrix} \mathbf{1}_p \\ \mathbf{0}_p \end{bmatrix} \qquad E_2 \triangleq \begin{bmatrix} \mathbf{0}_p \\ \mathbf{1}_p \end{bmatrix}$$

where $\mathbf{1}_p$ is the unit matrix of order p, $\mathbf{0}_p$ is the null matrix of order p, and successively premultiplying each constant chain matrix by the resulting matrix until an element of interest (which contains a variable parameter), a reference plane, or a termination is reached. The second type of analysis is the reverse analysis which consists of initializing a V matrix as either E_1 or E_2 and successively postmultiplying each constant matrix by the resulting matrix until an element of interest, a reference plane, or a termination is reached.

Consider the $2p$-port element shown in Fig. 1, possessing p input ports and p output ports. Its transmission matrix is given by

$$A \triangleq \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

where A_{11}, A_{12}, A_{21}, and A_{22} are $p \times p$ matrices. The input

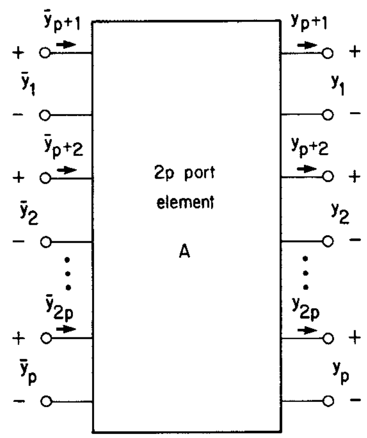

Fig. 1. A $2p$-port element.

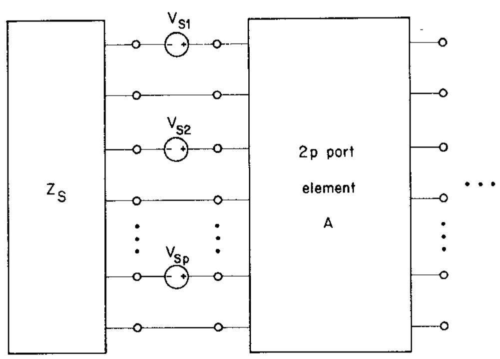

Fig. 2. Equivalent Thevenin voltages and impedance matrix for a subnetwork consisting of $2p$-port elements. Note that $V_{TH}=V_S$ and $Z_{TH}=Z_S$, which is not necessarily diagonal.

quantities in this case are

$$\bar{y}=\begin{bmatrix} \bar{y}_1 \\ \bar{y}_2 \\ \vdots \\ \bar{y}_p \\ \bar{y}_{p+1} \\ \bar{y}_{p+2} \\ \vdots \\ \bar{y}_{2p} \end{bmatrix}$$

and the output quantities are

$$y=\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_p \\ y_{p+1} \\ y_{p+2} \\ \vdots \\ y_{2p} \end{bmatrix}$$

where the elements with subscripts 1 to p denote voltages and from $p+1$ to $2p$ denote currents.

For the forward and reverse analyses, the matrices U_1, U_2, V_1, and V_2 are initialized such that

$$E_1 \Rightarrow U_1 \text{ or } V_1$$

$$E_2 \Rightarrow U_2 \text{ or } V_2.$$

We can now derive in an analogous manner to the derivation of [1, eq. (9)]

$$V_S=(\bar{U}_1^T+Z_S\bar{U}_2^T)A(V_1V_L+V_2(Y_LV_L-I_L)) \tag{1}$$

where $\bar{U}_1$, $\bar{U}_2$, V_1, and V_2 are the matrices obtained from forward and reverse analyses. V_S is the vector containing the p source voltages, V_L is the vector of load voltages, I_L is the vector of current sources at the loads (if any), Z_S is a diagonal matrix containing the source impedances, Y_L is the diagonal matrix containing the load admittances.

To evaluate the unknowns V_L, having obtained numerical values for (1), a system of p linear equations is solved.

To obtain the Thevenin voltages of the subnetwork on the left-hand side of the element A, we let $I_L=\mathbf{0}$ and $Y_L=\mathbf{0}$ in (1), which gives

$$V_S=(\bar{U}_1^T+Z_S\bar{U}_2^T)A\,V_1V_L=(Q_{11}+Z_SQ_{21})V_L \tag{2}$$

where

$$Q_{11}=\bar{U}_1^T A V_1 \tag{3}$$

and

$$Q_{21}=\bar{U}_2^T A V_1 \tag{4}$$

and from (2)

$$V_{TH}=V_L=(Q_{11}+Z_S Q_{21})^{-1} V_S. \tag{5}$$

We assume that the inverse of the matrix $(Q_{11}+Z_S Q_{21})$ exists. This is true when the network possesses a voltage-to-voltage transfer function. Consequently, this analysis fails (the matrix will be singular) if we have, for example, an element composed of current-controlled current sources. The output impedance matrix or the Thevenin impedance is obtained one column at a time by letting $V_S=0$, $Y_L=0$, and $I_L=0$ except I_{Li} (which is the current source at the load end for the ith port) which leads to

$$0=(Q_{11}+Z_S Q_{21})V_L-(Q_{12}+Z_S Q_{22})\begin{bmatrix}0\\ \vdots\\ I_{Li}\\ \vdots\\ 0\end{bmatrix} \tag{6}$$

where Q_{11} and Q_{21} are as defined in (3) and (4), respectively, and

$$Q_{12}=\bar{U}_1^T A V_2 \tag{7}$$

and

$$Q_{22}=\bar{U}_2^T A V_2. \tag{8}$$

Equation (6) can be written as

$$(Q_{11}+Z_S Q_{21})V_L=(Q_{12}+Z_S Q_{22})\begin{bmatrix}0\\ \vdots\\ I_{Li}\\ \vdots\\ 0\end{bmatrix}=C_i I_{Li} \tag{9}$$

where C_i is the ith column of the matrix $(Q_{12}+Z_S Q_{22})$. From (9) we get

$$V_L/I_{Li}=(Q_{11}+Z_S Q_{21})^{-1} C_i=\begin{bmatrix}Z_{TH_{1i}}\\ Z_{TH_{2i}}\\ \vdots\\ Z_{TH_{pi}}\end{bmatrix} \tag{10}$$

which is the ith column of the $p\times p$ Z_{TH} matrix. Fig. 2 shows the Z_{TH} and V_{TH} of the subnetwork preceding the element A. Similar formulas can be derived (analogous to [1, eqs. (13) and (14)]) for the input admittance matrix and the Norton current equivalent matrix.

As a special case when Z_S and Y_L are 0 or when they are considered as the first and last elements, respectively, (1) becomes

$$V_S=\bar{U}_1^T A V_1 V_L=Q_{11}V_L \tag{11}$$

so that the load voltages are given by

$$V_L=Q_{11}^{-1}V_S. \tag{12}$$

When A is perturbed to $A+\Delta A$, the new values for the load voltages $(V_L+\Delta V_L)$ can be obtained by six p^3 additional multiplications and the solution of a p-system of linear equations. Alternatively, the generalized Sherman–Morrison formula (Woodbury formula) [3] can be used to find $(Q_{11}+\Delta Q_{11})^{-1}$. Note that the reanalysis of the cascaded network is not performed. We use the results of only one analysis.

Differentiating (11) with respect to a parameter ϕ which appears in the matrix A only we get

$$0=\bar{U}_1^T\frac{\partial A}{\partial \phi}V_1 V_L+\bar{U}_1^T A V_1\frac{\partial V_L}{\partial \phi} \tag{13}$$

so that the sensitivity of the load voltages can be obtained from

$$\frac{\partial V_L}{\partial \phi}=-Q_{11}^{-1}\frac{\partial Q_{11}}{\partial \phi}V_L \tag{14}$$

where

$$\frac{\partial Q_{11}}{\partial \phi}=\bar{U}_1^T\frac{\partial A}{\partial \phi}V_1.$$

III. Numerical Example

The analysis and sensitivity evaluation of the response of a quasi-optical bandpass filter have been performed using the analysis approach described. The filter consists of three metallic (copper) wire grids in space with separations of 12.5 mm $(5\lambda/4)$. The equivalent circuit of the filter is shown in Fig. 3 [4]. The first and third gratings (in the x–y plane) have their wires parallel to the x axis, while the middle grating has the wires oriented at an angle ϕ with respect to the x axis. The circuits $R(\phi)$ and $R(-\phi)$ are used to connect the middle grating with the adjacent local coordinates (the equivalent circuit of the filter is based on the local coordinate concept [4]). The free space between the gratings is represented by the uncoupled transmission lines (with lengths equal to the separation between the gratings) as shown in Fig. 3. The parameters B_a, B_b, X_a, and X_b can be found in [5] and R_p and X_p are from [4]. The dimensions of the gratings are given in Fig. 4. The dielectric sheets supporting the metal gratings were not considered in our analysis. The filter is excited by a source representing an incident wave linearly polarized in the direction perpendicular to the first grating (i.e., polarized in the y direction). The transmitted wave is represented by the output voltage at port 3.

The insertion loss of the filter (the center frequency f_0 is equal to 30 GHz) is shown in Fig. 5 for various angles ϕ.

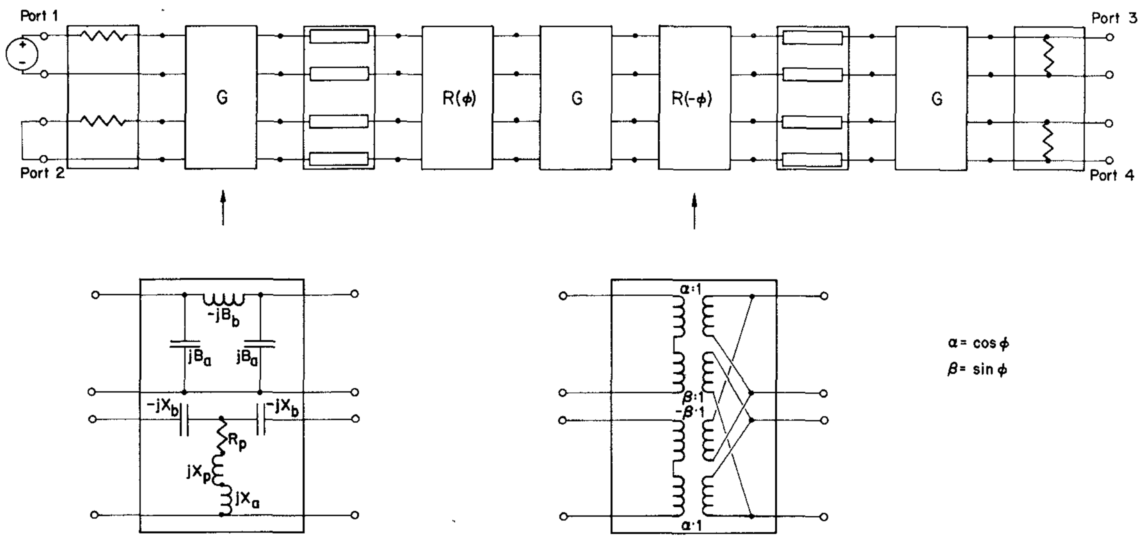

Fig. 3. Equivalent circuit of the quasi-optical filter [4].

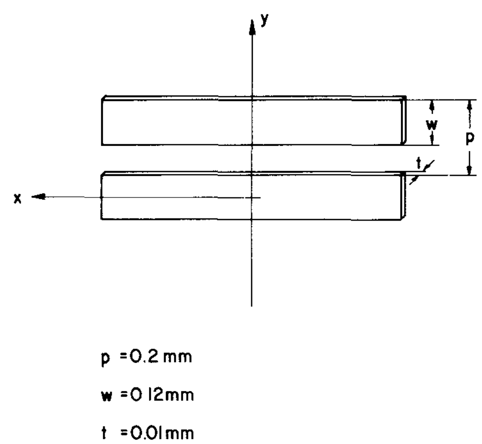

Fig. 4. Physical dimensions of the wire grid.

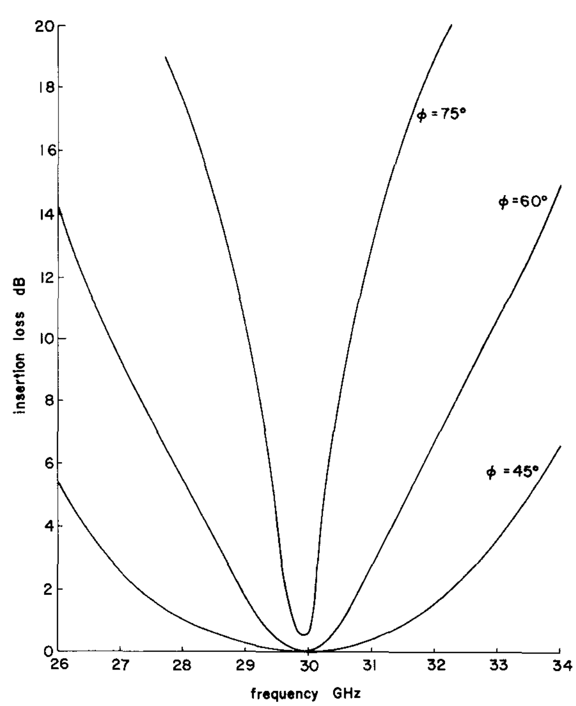

Fig. 5. Insertion loss of the filter for different values of ϕ.

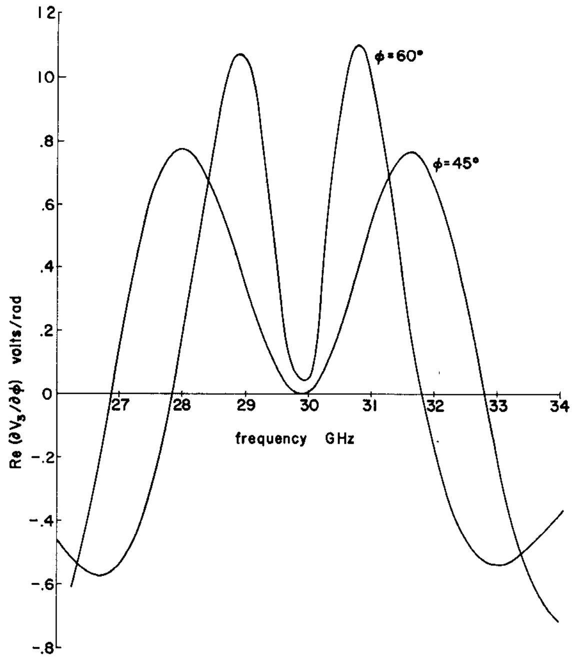

Fig. 6. Real part of $\partial V_3/\partial\phi$ at $\phi=45°$ and $\phi=60°$.

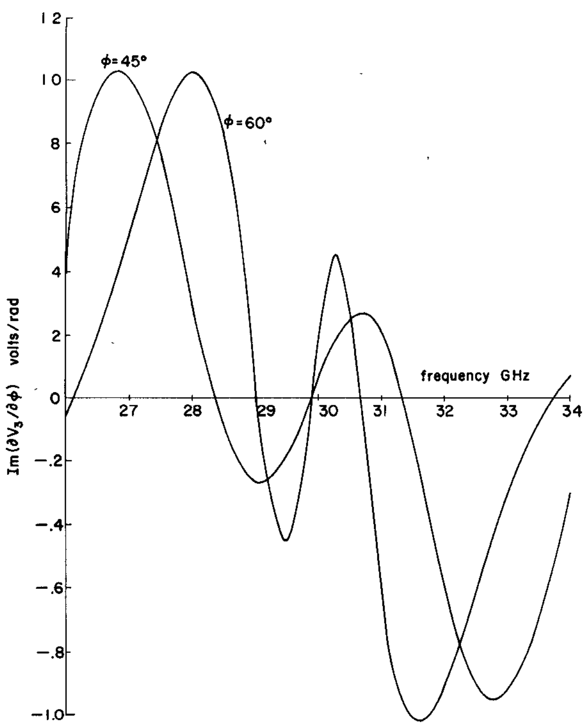

Fig. 7. Imaginary part of $\partial V_3/\partial\phi$ at $\phi=45°$ and $\phi=60°$.

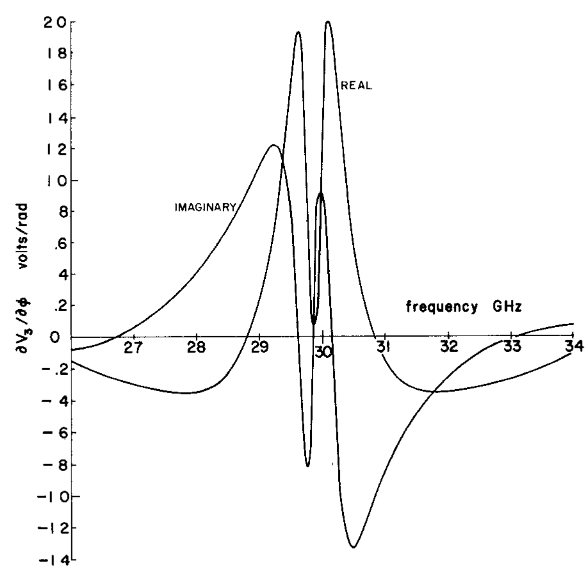

Fig. 8. Real and imaginary parts of $\partial V_3/\partial\phi$ at $\phi=75°$.

The exact sensitivity of the voltage at port 3 with respect to ϕ is plotted in Figs. 6–8 for different values of ϕ.

IV. Conclusions

The use of this analysis approach avoids the need for reanalyzing the cascaded networks to evaluate large change sensitivities. It also facilitates the evaluation of first-order sensitivities of the response with respect to variable design parameters without defining and analyzing any additional network (adjoint network). These advantages lead to a considerable saving in computational time and effort.

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